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KIRIRI WOMENS' UNIVERSITY OF SCIENCE AND TECHNOLOGY
UNIVERSITY EXAMINATIONS, 2022/2023 ACADEMIC YEAR
FOURTH YEAR, SECOND SEMESTER EXAMINATION
FOR THE DEGREE OF BACHELOR OF SCIENCE
(MATHEMATICS)

KMA 414 - MEASURE THEORY AND PROBABILITY

Date: 19th April 2022
Time: 8.30am-10.30am

INSTRUCTIONS TO CANDIDATES
ANSWER QUESTION ONE (COMPULSORY) AND ANY OTHER TWO QUESTIONS

QUESTION ONE (30 MARKS)

a) Define;

i) A set. (1 mark)
ii) A Borel field. (1 mark)
iii) An indicator function. (1 mark)
iv) Convergence in distribution. (1 mark)

b) Let A be a set with n elements and P^A be the power set of A. Show that the number of subsets of A contained in P^A is 2^n. (5 marks)

c) Let $A_1, A_2, \ldots, A_n$ be the sets which are not necessarily disjoint. Show that there exist disjoint sets $B_1, B_2, \ldots, B_n$ which are disjoint such that

$$\bigcup_{i=1}^{n} A_i = \sum_{i=1}^{n} B_i \text{. Hence show that } P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(B_i).$$

(6 marks)

d) Define a measure. Show that probability is a measure. (5 marks)

e) Show that the necessary and sufficient condition for a random variable X to converge in probability to zero is that $E\left[\frac{|X|}{1+|X|}\right]$. (5 marks)

f) Let $X \sim N(50, 100)$. Use Chebyshevs inequality to find the lower bound for

i) $P(|\overline{X} - \mu| > 5)$ where $\overline{X}$ was obtained from a sample of size 20. (3 marks)

ii) $P(|X - \mu| > 20)$. (2 marks)

QUESTION TWO (20 MARKS)

a) Let A be the set $A = [X: |X| \geq a]$ and g be a measurable Borel function on $\mathcal{R}$. Prove the basic inequality that

$$\frac{E[g(x)]-g(a)}{Sup.\{g(x)\}} \leq P(A) \leq \frac{E[g(x)]}{g(a)} \qquad \text{(10 marks)}$$

b) State the Markov's inequality, hence use basic inequality in a) to prove it. (4 marks)

c) State the Chebyshevs inequality, hence use Markov chain in b) to prove it. (6 marks)

QUESTION THREE (20 MARKS)

a) What is a field? (2 marks)

b) Prove that a monotone field is a $\sigma -$ field and vice-versa. (6 marks)

c) Prove

i) Holder's inequality $E[XY] \leq E^{\frac{1}{r}}|X|^r E^{\frac{1}{s}}|X|^s$, hence show that $E[XY] \leq \sqrt{E\,[X]^2\, E\,[Y]^2}$.

(6 marks)

ii) $E^{\frac{1}{r}}|X+Y|^r \leq E^{\frac{1}{r}}|X|^r + E^{\frac{1}{r}}|Y|^r$. (6 marks)

QUESTION FOUR (20 MARKS)

a) Discuss four properties of a cumulative distribution function. (4 marks)

b) Describe the axiomatic concept of probability. (5 marks)

c) Let $X_1, X_2, \dots, X_n$ be i.i.d binomial random variables with probability distribution Bin $(n, \frac{\lambda}{n})$. Show that for large n, the distribution converges to Pois (λ). (7 marks)

d) State, without prove, the Borel-Cantelli Lemma. (4 marks)

QUESTION FIVE (20 MARKS)

a) Let $I_A = \begin{cases} 1, & i \in A \\ 0, & i \notin A \end{cases}$ and $I_B = \begin{cases} 1, & i \in B \\ 0, & i \notin B \end{cases}$ be the indicator functions of sets A and B respectively.

Show that

i) Whenever $A \subset B,\ I_B \geq I_A$. (2 marks)

ii) $I_{AB} = I_A + I_B - I_{A\cup B}$. (3 marks)

b) Let X and Y be two random variables, $I_A = \begin{cases} 1, & i \in A \\ 0, & i \notin A \end{cases}$ $I_B = \begin{cases} 1, & i \in B \\ 0, & i \notin B \end{cases}$ $and\ I_{AB} = \begin{cases} 1, & i \in A \cap B \\ 0, & i \notin A \cap B \end{cases}$.

Prove that $E(X \mp Y) = E(X) \mp E(Y)$. (5 marks)

c) Differentiate between a monotone increasing and monotone decreasing sequence. Hence determine whether the sequences $A_n = \{w: 2 - \frac{2}{n} \leq w \leq 5 + \frac{1}{n},\ n \in \mathbb{R}\}$ is a monotone increasing or monotone decreasing and state its limit. (6 marks)

d) Show that inverse mapping preserves the following set relation.

$$X^{-1}\left(\bigcap_k B_k'\right) = \bigcap_k X^{-1}(B_k') \qquad \text{(4 marks)}$$