Kasarani Campus
Off Thika Road
Tel. 2042692 / 3
P. O. Box 49274, 00100
NAIROBI
Westlands Campus
Pamstech House
Woodvale Grove
Tel. 4442212
Fax: 4444175

KIRIRI WOMENS' UNIVERSITY OF SCIENCE AND TECHNOLOGY
UNIVERSITY EXAMINATION, 2023/2024 ACADEMIC YEAR
FOURTH YEAR, SECOND SEMESTER EXAMINATION
FOR THE DEGREE OF BACHELOR OF SCIENCE IN MATHEMATICS
KMA 410- MULTIVARIATE I

Date: 14TH APRIL 2023
Time: 11:30 AM-1:30 PM

INSTRUCTIONS TO CANDIDATES
ANSWER QUESTION ONE (COMPULSORY) AND ANY OTHER TWO QUESTIONS

QUESTION ONE (30 MARKS)

a) Consider the following sample data matrix with three variables

$$X = \begin{pmatrix} 9 & 12 & 3 \\ 2 & 8 & 4 \\ 6 & 6 & 0 \\ 5 & 4 & 2 \\ 8 & 10 & 1 \end{pmatrix}$$

i) Obtain the sample covariance matrix (4 Marks)

ii) Use the obtained covariance matrix to find variance of $2X_1 + X_2 - 3X_3$ (3 Marks)

b) Explain the following concepts and their significance as used in multivariate analysis

i) Central Limit Theorem (2 Marks)

ii) Law of Large Numbers (2 Marks)

c) Consider the random vector $\underline{X} = (X_1, X_2, X_3)'$ with a pdf given by;

$$f(\underline{x}) = \begin{cases} kx_1x_2x_3, & 0 < x_1, x_2 < 1, \quad 0 < x_3 < 3 \\ 0, & otherwise \end{cases}$$

i) Find the value of the constant k (3 Marks)

ii) Obtain the marginal distributions of $X_1, X_2,$ and X_3 and check for independence of $X_1, X_2,$ and X_3 (5 Marks)

d) The variates $\underline{X}' = (X_1, X_2, X_3)$ and $\underline{Y}' = (Y_1, Y_2, Y_3)$ are distributed independently according to the trivariate normal populations with respective parameters.

$$\mu_1 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \quad \sum{}_1 = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \quad \mu_2 = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}, \quad \sum{}_2 = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 4 & 2 \\ 0 & 2 & 4 \end{bmatrix},$$

Determine the distribution of

i) $Z = \underline{X}' - \underline{Y}'$ (3 Marks)

ii) $Q = \underline{X}' + \underline{Y}'$ (3 Marks)

e) Suppose $Y \sim N_3(\underline{\mu}, \sum)$ where $\mu = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$ and $\sum = \begin{bmatrix} 4 & -3 & 0 \\ -3 & 6 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

By appropriate partitioning, examine the independence between (Y_1, Y_3) and Y_2

(5 Marks)

QUESTION TWO (20 MARKS)

a) Consider a bivariate normal distribution with $\mu_1 = 0,\ \mu_2 = 2,\ \sigma_{11} = \sigma_{22} = 4$ and $\rho = 0.75$.

i) Obtain the covariance matrix and write the bivariate normal density as a quadratic function of X_1 and X_2 (5 Marks)

ii) Hence, obtain the conditional distribution of X_2 given X_1 (4 Marks)

b) Let the random variables X_1 and X_2 have a joint probability generating function

$$P(S_1, S_2) = \exp[\lambda_1(S_1 - 1) + \lambda_2(S_2 - 1) + \lambda_3(S_1 S_2 - 1)]$$

Find;

i) Mean and variance of X_1 (4 Marks)

ii) The covariance of X_1 and X_2 (4 Marks)

iii) The correlation coefficient between X_1 and X_2 (3 Marks)

QUESTION THREE (20 MARKS)

a) Define a characteristic function of a random vector. (3 Marks)

b) Obtain the characteristic function of a p-dimensional random vector $\underline{X} \sim N_p(\underline{\mu}, \Sigma)$

(9 Marks)

c) Define $Y = a'\underline{X},\quad a \neq 0$, where a is a vector of constants. Use the characteristic function technique to show that $Y \sim N_p(a'\underline{\mu}, a'\Sigma a)$ (4 Marks)

d) Let $\underline{X} \sim N_3(\underline{\mu}, \Sigma)$, where $\underline{\mu}' = (2, -1, 3)'$ and $\Sigma = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix}$. Find the distribution of

$$Z = \begin{pmatrix} X_1 - X_2 + X_3 \\ 2X_1 + X_2 - X_3 \end{pmatrix}$$

(4 Marks)

QUESTION FOUR (20 MARKS)

a) Let $\underline{X}$ be a p-variate random vector with mean vector $\underline{\mu}$ and variance-covariance matrix $\Sigma = ((\sigma_{ij})i, j = 1, 2, ..., p)$. Let A be a symmetric matrix such that the quadratic form on $\underline{X}$ is given by $Q = \underline{X}' A\, \underline{X}$. Show that $E\left(\underline{X}' A\, \underline{X}\right) = Trace(A\Sigma) + \underline{\mu}' A\, \underline{\mu}$ (6 Marks)

b) Define a quadratic form $M = \sum_{i=1}^{n-1} (X_i - X_{i+1})^2$. Obtain the unbiased estimator of σ^2 using expectation of the quadratic form. (10 Marks)

c) Given $\underline{X} \sim N_p(\underline{\mu}, \Sigma)$, and $f(\underline{X}) = C \exp\left\{-\frac{1}{2} Q\right\}$ where Q has the usual meaning. Show that $E(Q) = p$ (4 Marks)

QUESTION FIVE (20 MARKS)

a) Suppose X and Y are random variables with joint density function

$$f(x,y)=\begin{cases} x+y, & 0\le x,y\le 1 \\ 0, & elsewhere \end{cases}$$

Find the correlation matrix of $Z=(X\ ,Y)'$. Hence or otherwise, comment of the relationship between X and Y. (8 Marks)

b) Given that $X=(X_1,\ X_2,\ X_3)'$ with mean vector $\underline{\mu}'=(-1,1,2)'$ and $\Sigma=\begin{bmatrix} 4 & -2 & 1 \\ -2 & 17 & 3 \\ 1 & 3 & 2 \end{bmatrix}$

Find the conditional distribution of X_2 given $(X_1,\ X_3)=(0\ ,\ \ 3)'$ (8 Marks)

c) Let $X_1\sim N(\mu_1,\sigma_1^2)$ and $X_2\sim N(\mu_2,\sigma_2^2)$ and given that X_1 and X_2 are independent random variables. Find the distribution of $Y=X_1-X_2$ using the moment generating function (mgf) technique. (4 Marks)