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KIRIRI WOMENS' UNIVERSITY OF SCIENCE AND TECHNOLOGY
UNIVERSITY EXAMINATIONS, 2024/2025 ACADEMIC YEAR
FIRST YEAR, SECOND SEMESTER EXAMINATION
FOR MASTER OF SCIENCE IN APPLIED STATISTICS AND DATA ANALYTICS
KMA 5106: MODERN STATISTICAL INFERENCE

DATE: 31ST JANUARY, 2025
TIME: 9:00 AM – 12:00 PM

INSTRUCTIONS TO CANDIDATES
ANSWER QUESTION ONE (COMPULSORY) AND ANY OTHER THREE QUESTIONS

QUESTION ONE: COMPULSORY (40 MARKS)

a) Discuss the two main paradigms of statistical inference clearly highlighting their essential features and potential conflicts. **[4 Marks]**

b) Explain three types of prior distribution as used in Bayesian inference. **[6 Marks]**

c) Explain two approaches used to strike a balance between the probabilities of type I and type II errors. **[4 Marks]**

d) Let $\underline{Y} = (Y_1, \dots, Y_n)^{\mathrm{T}}$ be a random sample from a Gamma distribution Gamma(λ, α) with the following pdf

$$f(y \mid \lambda, \alpha) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} y^{\alpha-1} e^{-\lambda y}, \text{ for } y > 0.$$

(i) Show that the distribution belongs to the exponential family. **[5 Marks]**

(ii) Identify the joint complete sufficient statistics for $(\lambda, \alpha)^{\mathrm{T}}$. **[2 Marks]**

e) Discuss two main properties of maximum likelihood estimators. **[4 Marks]**

f) An urn contains θ black balls and $N - \theta$ white balls. A sample of n balls is selected without replacement. Let Y denote the number of black balls in the sample. Find the method of moments estimator of θ. Estimate the number of black balls in the urn when $N = 12, n = 6$ and you observed $y = 5$. **[5 Marks]**

g) Suppose that $X_1, X_2, \cdots, X_n$ are i.i.d. random variables on the interval [0,1] with the density function

$$f(x \mid \alpha) = \frac{\Gamma(2\alpha)}{\Gamma(\alpha)^2} [x(1-x)]^{\alpha-1}$$

where $\alpha > 0$ is a parameter to be estimated from the sample. Find a sufficient statistic for α. **[5 Marks]**

h) Suppose that $X_1, \ldots, X_n$ form a random sample from the Poisson distribution with mean $\theta > 0$, where θ is unknown. Suppose that the prior distribution of θ is $\text{Gamma}(\alpha, \beta)$, where $\alpha, \beta > 0$. Derive the posterior distribution of θ. **[5 Marks]**

QUESTION TWO: (20 MARKS)

a) Show that for any estimator T_n of $g(\theta)$, the mean squared error, $MSE(T_n, g(\theta))$ can be decomposed into variance and bias components. **[4 Marks]**

b) Suppose that the distribution of the lifetime of fluorescent tubes of a certain type is the exponential distribution with parameter θ. Suppose that $X_1, \ldots, X_n$ is a random sample of lamps of this type. Also suppose that $\theta \sim \text{Gamma}(\alpha, \beta)$, for known α, β. Derive the posterior distribution of θ. **[5 Marks]**

c) The Pareto distribution is commonly used in economics as a model for a density function with a slowly decaying tail:

$$f(x \mid x_0, \theta) = \theta x_0^{\theta} x^{-\theta-1}, \quad x \geq x_0, \quad \theta > 1$$

Assume that $x_0 > 0$ is given and that $X_1, X_2, \cdots, X_n$ is an i.i.d. sample. Find the asymptotic distribution of the mle of θ. **[4 Marks]**

d) Let $X_1, \cdots, X_n$ denote a random sample from a Poisson distribution that has mean $\lambda > 0$. Derive an approximate $100(1-\alpha)\%$ confidence interval for λ. **[3 Marks]**

e) Show that $[\sum x^2 \sum x]^T$ is a minimal sufficient statistic for $[\mu, \sigma^2]^T$, where $X_i \overset{iid}{\sim} N(\mu, \sigma^2)$. **[4 Marks]**

QUESTION THREE: (20 MARKS)

a) Suppose that $Y_1, Y_2, \ldots, Y_n$ is an independent sample from the distribution of a random variable Y, where $Y \sim \text{Weibull}(\beta)$ with the pdf given by

$$f(y \mid \beta) = \begin{cases} \frac{1}{\beta} 2y e^{-\frac{y^2}{\beta}}, & y > 0, \beta > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

(i) Find a minimum sufficient statistic for β. **[4 Marks]**

(ii) Knowing that $E(Y^2) = \beta$ find an unbiased minimum sufficient statistic for β. **[3 Marks]**

b) Suppose that $X_1, \ldots, X_n$ form a random sample from $N(\theta, \sigma^2)$, where θ is unknown and the value of the variance $\sigma^2 > 0$ is known. Suppose that $\Theta \sim N(\mu_0, v_0^2)$. Show that . **[7 Marks]**

$$\Theta \mid X_1 = x_1, \ldots, X_n = x_n \sim N(\mu_1, v_1^2),$$

where

$$\mu_1 = \frac{\sigma^2 \mu_0 + n v_0^2 \bar{x}_n}{\sigma^2 + n v_0^2} \quad \text{and} \quad v_1^2 = \frac{\sigma^2 v_0^2}{\sigma^2 + n v_0^2}$$

c) Define $p - value$ and explain its usage in statistical hypothesis testing. **[2 Marks]**

d) Suppose that $X_1, \ldots, X_n$ are i.i.d Uniform $(0, \theta)$, where $\theta > 0$ is unknown. Suppose that we are interested in the following hypotheses:

$$H_0: 3 \leq \theta \leq 4, \quad \text{versus} \quad H_1: \theta < 3, \text{ or } \theta > 4.$$

Suppose that we use a test δ given by the critical region

$$S_1 = \{x \in \mathbb{R}^n : x_{(n)} \leq 2.9 \text{ or } x_{(n)} \geq 4\}.$$

Find the power function $\pi(\theta \mid \delta)$? **[4 Marks]**

QUESTION FOUR: (20 MARKS)

a) Discuss the following terms as used in Bayesian inference
   (i) Prior distribution. **[2 Marks]**
   (ii) Posterior distribution. **[2 Marks]**
   (iii) Bayes estimator. **[2 Marks]**

b) Suppose that $Y_1, Y_2, \ldots, Y_n$ is an independent sample from the distribution of a random variable Y, where $Y \sim \text{Poisson}(\lambda)$. Show that $T(\underline{Y}) = \sum_{i=1}^{n} Y_i$ is a complete sufficient statistic for λ. **[5 Marks]**

c) Discuss the following terms as used in statistical inference
   (i) Minimal sufficient statistic. **[2 Marks]**
   (ii) Ancillary statistic. **[2 Marks]**

d) Suppose random variable X has a Bernoulli distribution for which the parameter θ is unknown $(0 < \theta < 1)$. Determine the Fisher information $I(\theta)$ in X. **[5 Marks]**

QUESTION FIVE: (20 MARKS)

a) Let $\underline{Y} = (Y_1, \ldots, Y_n)^{\mathrm{T}}$ be a random sample from the distribution with the pdf given by

$$f(y \mid \theta) = \begin{cases} \frac{2}{\theta^2}(\theta - y), & y \in [0, \theta], \\ 0, & \text{elsewhere.} \end{cases}$$

   (i) Find an estimator of θ using the method of moments. **[3 Marks]**
   (ii) Calculate its bias and variance and check if it is a consistent estimator of θ .
   [6 Marks]

b) Suppose that Y is a single observation from a population with beta distribution

$$f_Y(y) = \theta y^{\theta - 1}, \quad 0 \leq y \leq 1$$

   where $\theta > 0$ is a parameter.
   (i) For testing $H_0: \theta \leq 1$ against $H_1: \theta > 1$, find the significance level and the power function of the test that rejects H_0 if $Y > 1/2$. **[4 Marks]**
   (ii) Find the most powerful level α test of $H_0: \theta = 1$ against $H_1: \theta = 2$. **[4 marks]**
   (iii) Is there a uniformly most powerful test of $H_0: \theta = 1$ against $H_1: \theta > 1$? Justify your answer. **[3 Marks]**